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Analytic Modelling of the Long-term Evolution of Orbital Debris

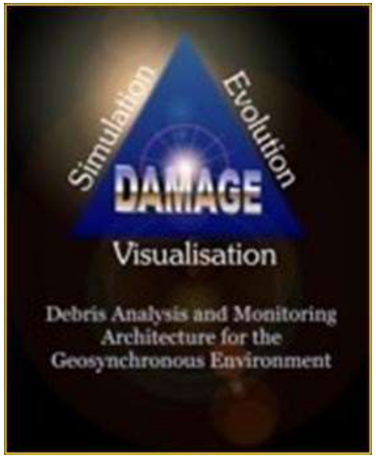
Colin McInnes

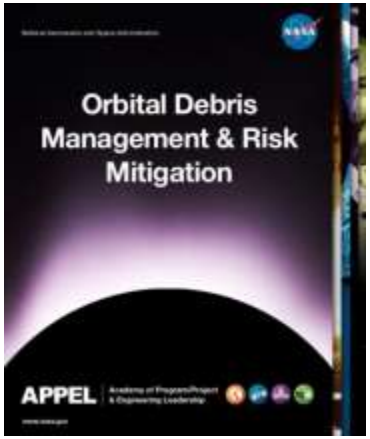
frontier research on visionary space systems

Debris modelling

- **Debris evolution modelling often uses numerical simulation (eg. MASTER)**
- **Can simple analytical models provide useful insights into debris evolution?**
- **Present two simple models for population dynamics using ODE and PDE**

MASTER 2009 esa





Particle-in-a-box model

- **Simple model to capture key process in evolution of orbit debris population**
- **Model no. of debris objects *N* with non-linear first order evolution equation**
- **Define debris deposition rate *A*, removal rate *B* and collisional growth rate *C***

$$\boxed{\frac{dN}{dt} = A + BN + CN^2}$$

$$\dot{N} = 0 \longrightarrow N_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C}$$

Talent, D: 'Analytical model for orbital debris environmental management'. J. Spacecraft and Rockets, Vol. 29, No. 4, July-August 1992

Stable and unstable regiemes

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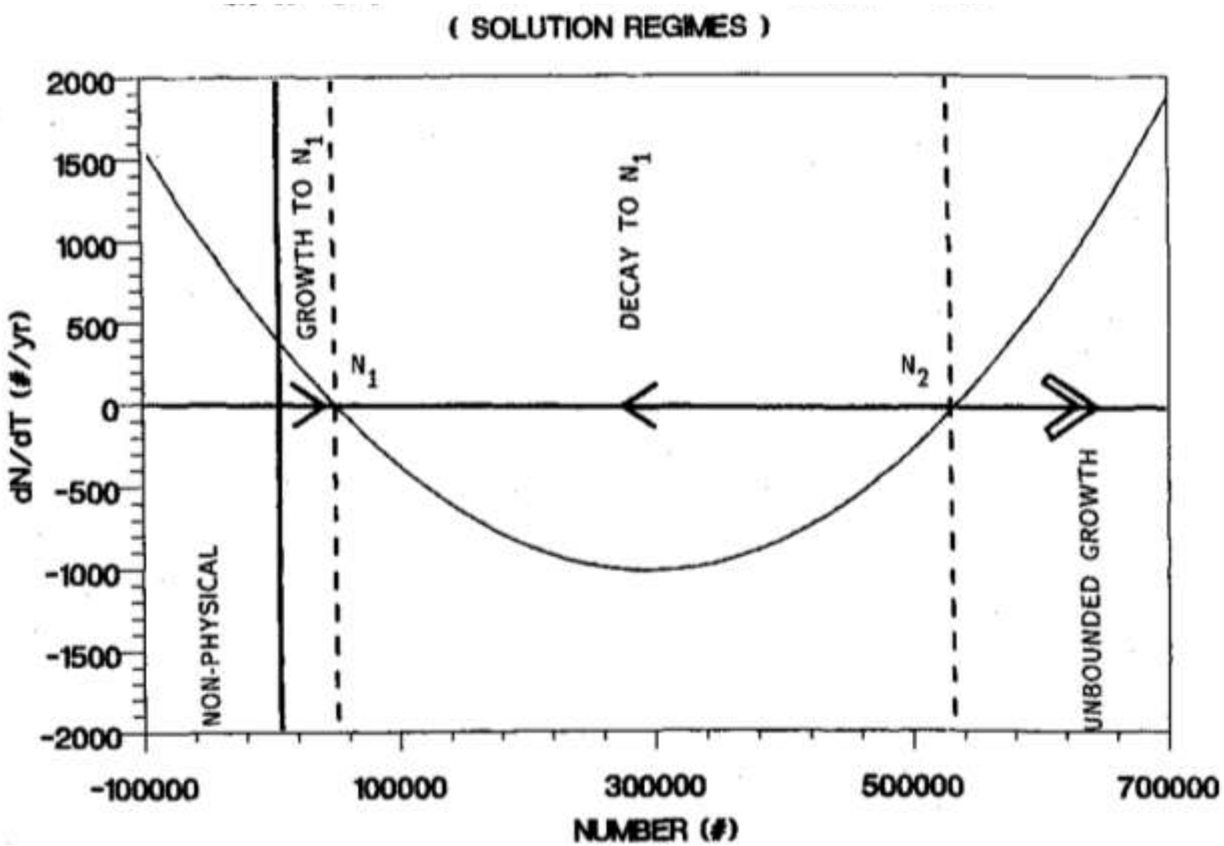
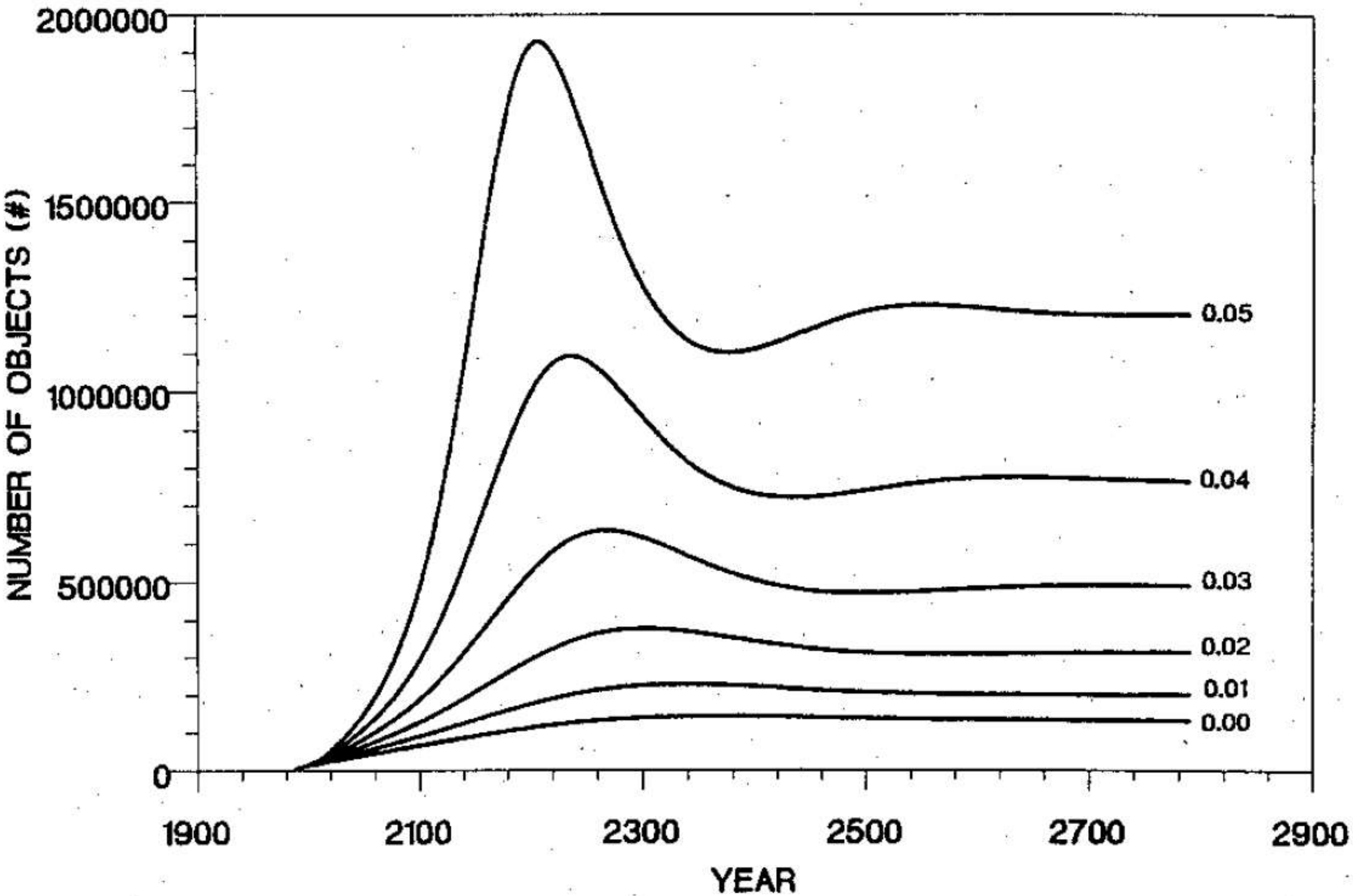
- **Equilibrium condition yields quadratic with 2 distinct roots (debris end states)**
- **Discriminant of quadratic is a measure of difference between sinks/sources**
- **Therefore have 3 possible conditions . . .**

$$
\begin{aligned}
q &= B^2 - 4AC \\
&= Sinks - Sources
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
&q > 0 \Rightarrow Sinks > Sources \Rightarrow STABLE \\
&q = 0 \Rightarrow Sinks = Sources \Rightarrow THRESHOLD \\
&q < 0 \Rightarrow Sinks < Sources \Rightarrow UNSTABLE
\end{aligned}
}
$$



$$q > 0 \Rightarrow Sinks > Sources \Rightarrow STABLE$$


2000000
1500000
1000000
500000
0
NUMBER OF OBJECTS (#)
1900
2100
2300
2500
2700
2900
YEAR
0.05
0.04
0.03
0.02
0.01
0.00

Growth
0-5%

Continuum PDE model

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- **Assume that debris population is a continuum (really a collisionless fluid)**
- **Debris population is therefore described by the debris number density *n***
- **For continuum problems can use the tools of *partial differential equations***

McInnes, C.R.: 'An analytical model for the catastrophic production of orbital debris', European Space Agency Journal, Vol. 17, No. 4, pp. 293-305, 1993.

McInnes, C.R.: 'A simple analytic model of the long term evolution of nanosatellite constellations', Journal of Guidance, Control and Dynamics, Vol. 23, No. 2, pp. 332-338, 2000.

JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 2, March–April 2000

Simple Analytic Model of the Long-Term Evolution of Nanosatellite Constellations

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The long-term evolution of a large constellation of nanosatellites is investigated using closed-form analytical methods. A set of partial differential equations is derived, the solution to which provides the evolution of the mean spatial number density of the constellation under the action of air drag, on-orbit satellite failures, and the deposition of new satellites into the constellation. These solutions give insight into the global dynamics and long-term evolution of large constellations of nanosatellites and display some interesting physical features. In particular, asymptotic solutions provided the steady-state distribution of nanosatellites and an estimate of the required rate of deposition of new nanosatellites to maintain the constellation.

Population dynamics

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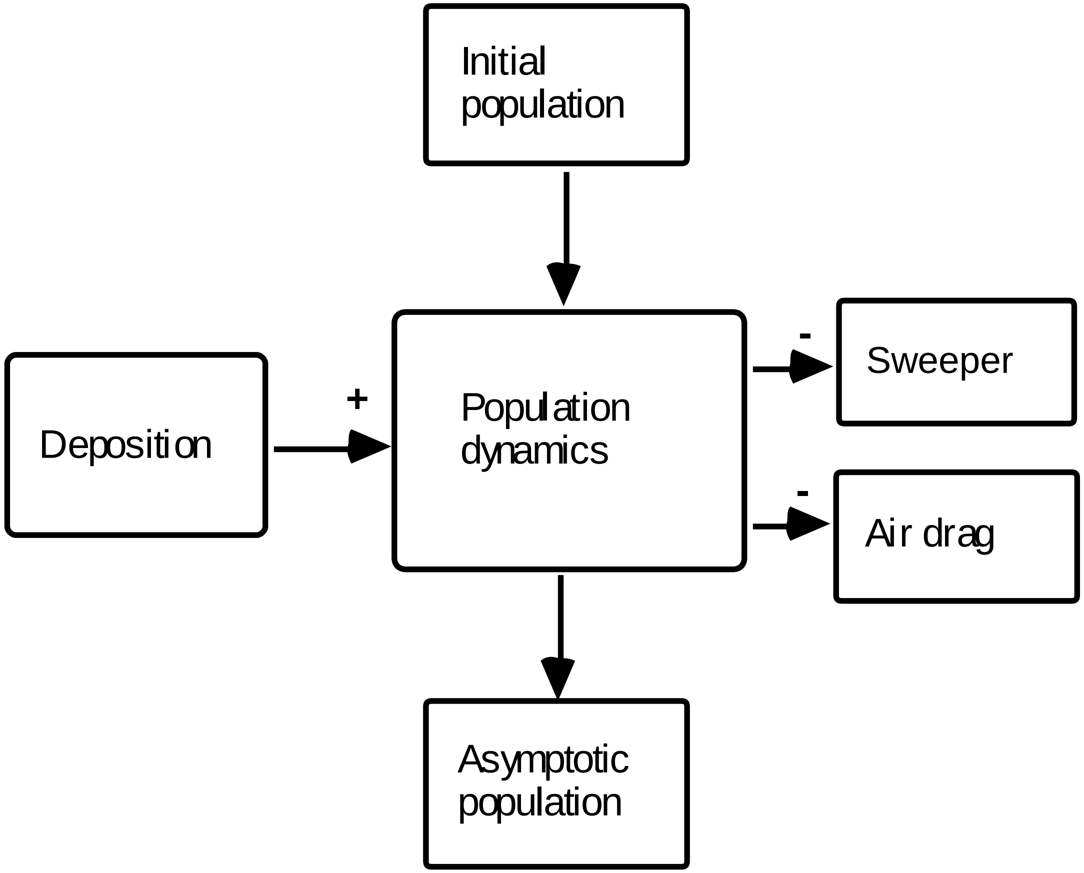
- Assume the debris density ***n*** is now a function of orbit radius ***r*** and time ***t***
- Assume there exist **source terms** (new objects) and **sink terms** (sweepers)
- Also, the inflow due to air drag a radial speed v_r (*r*) which is a function of ***r***

$$\frac{\partial n(r,t)}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r(r) n(r,t)\right)=\dot{n}^{+}(r,t)-\dot{n}^{-}(r,t)$$

We'll derive this shortly. . . . .


Initial population
Deposition
+
Population dynamics
-
Sweeper
-
Air drag
Asymptotic population

Evolution timescales

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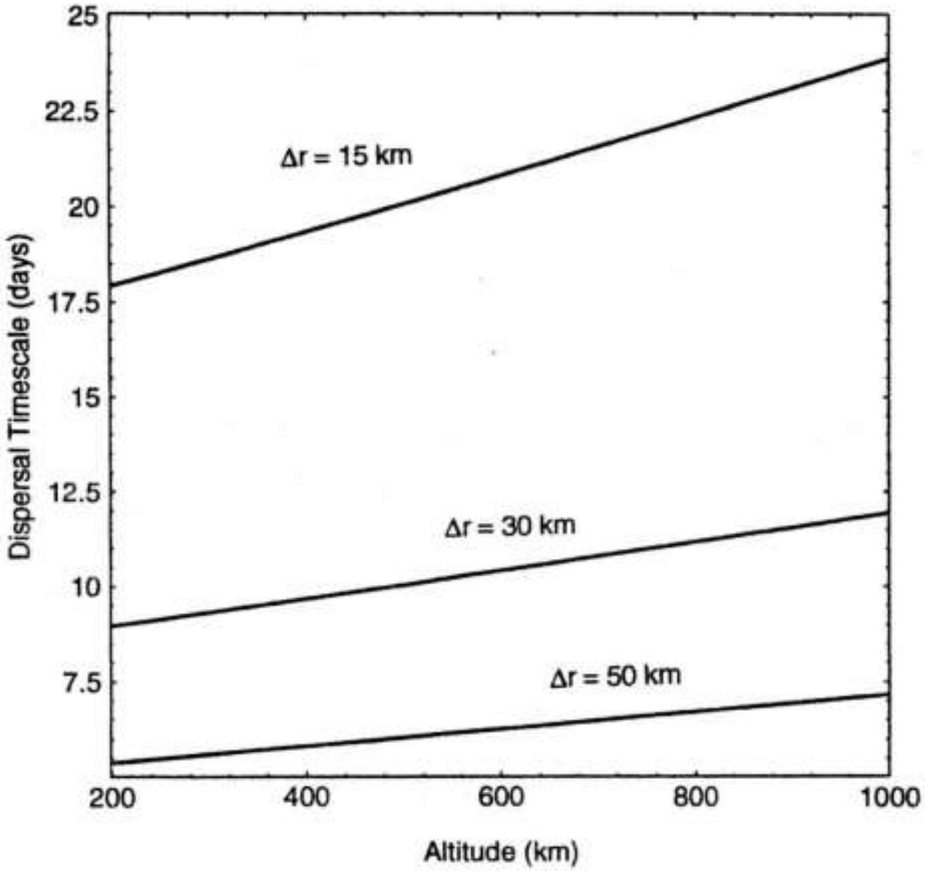
- **Assuming that debris population described as a function of orbit radius only**
- **Could model entire population as a symmetric sphere with radial structure**
- **Could model fragmentation events after a uniform torus has been formed**

$$\omega = \sqrt{\frac{\mu}{r^3}}$$

$$\Delta\omega = \frac{\partial\omega}{\partial r}\Delta r$$

$$T_\theta |\Delta\omega| = 2\pi$$

$$T_\theta \sim \frac{4\pi}{3\sqrt{\mu}} \frac{r^{5/2}}{\Delta r}$$



25
22.5
20
17.5
15
12.5
10
7.5
Dispersal Timescale (days)
Δr = 15 km
Δr = 30 km
Δr = 50 km
200
400
600
800
1000
Altitude (km)

Particle dynamics

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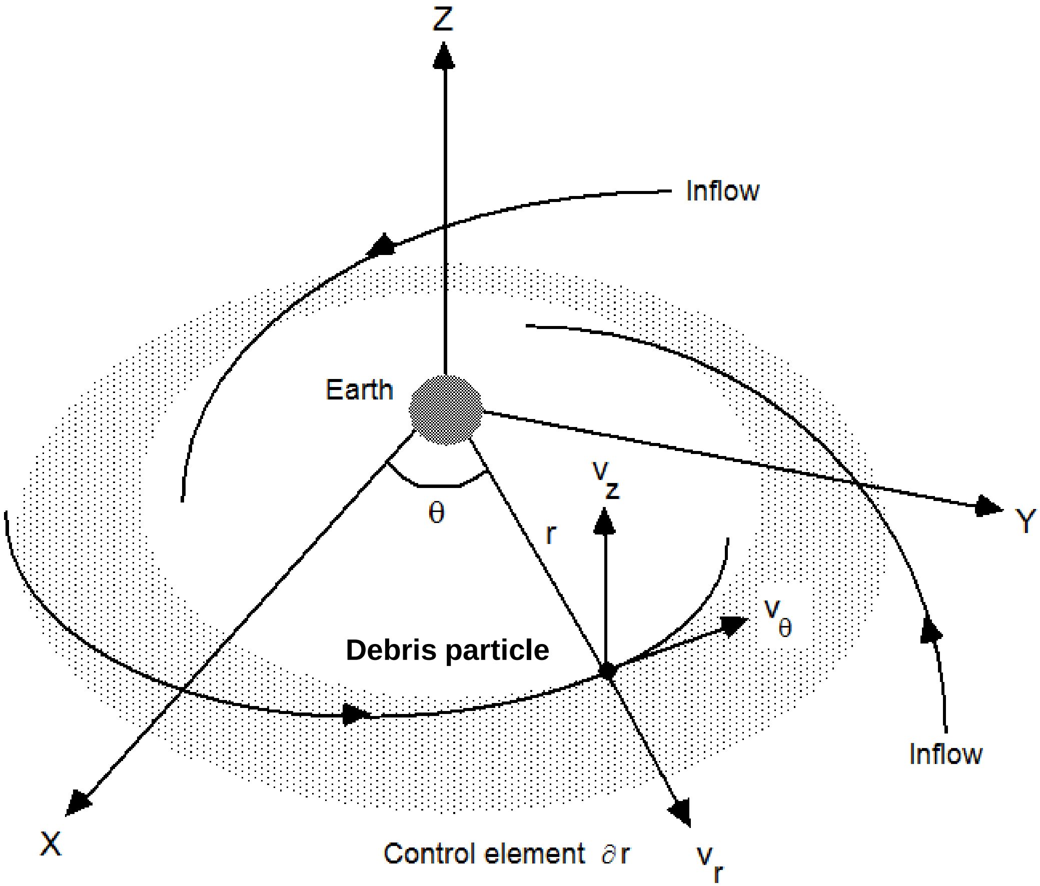
- Assume that motion of debris is quasi-circular so local Keplerian motion
- For 2D disk assume that motion is entirely planar (the torus is flat . . . . .)
- Radial inflow due to air drag a function of orbit radius *r* and ballistic coeff *B*

$$v_r = -f(r, B)$$

$$v_\theta = \sqrt{\frac{\mu}{r^3}}$$

$$v_z \sim 0$$

Inflow of debris due to air drag


Z
Inflow
Earth
θ
r
v_z
Y
v_θ
Debris particle
Inflow
X
Control element ∂r
v_r

Air drag

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- Assume isothermal exponential atmosphere of density *r* and scale height *H*
- Determine inflow speed from the work done by air drag on the debris particle
- For quasi-circular motion obtain inflow speed of debris (radial shearing . . .)

$$a = \frac{1}{2} B\rho(r)v^2 \quad , \quad \rho(r) = \rho_o \exp\left(-\frac{r-R}{H}\right)$$

$$v_r(r) = -\sqrt{\mu r}\, B\rho_o \exp\left(-\frac{r-R}{H}\right)$$

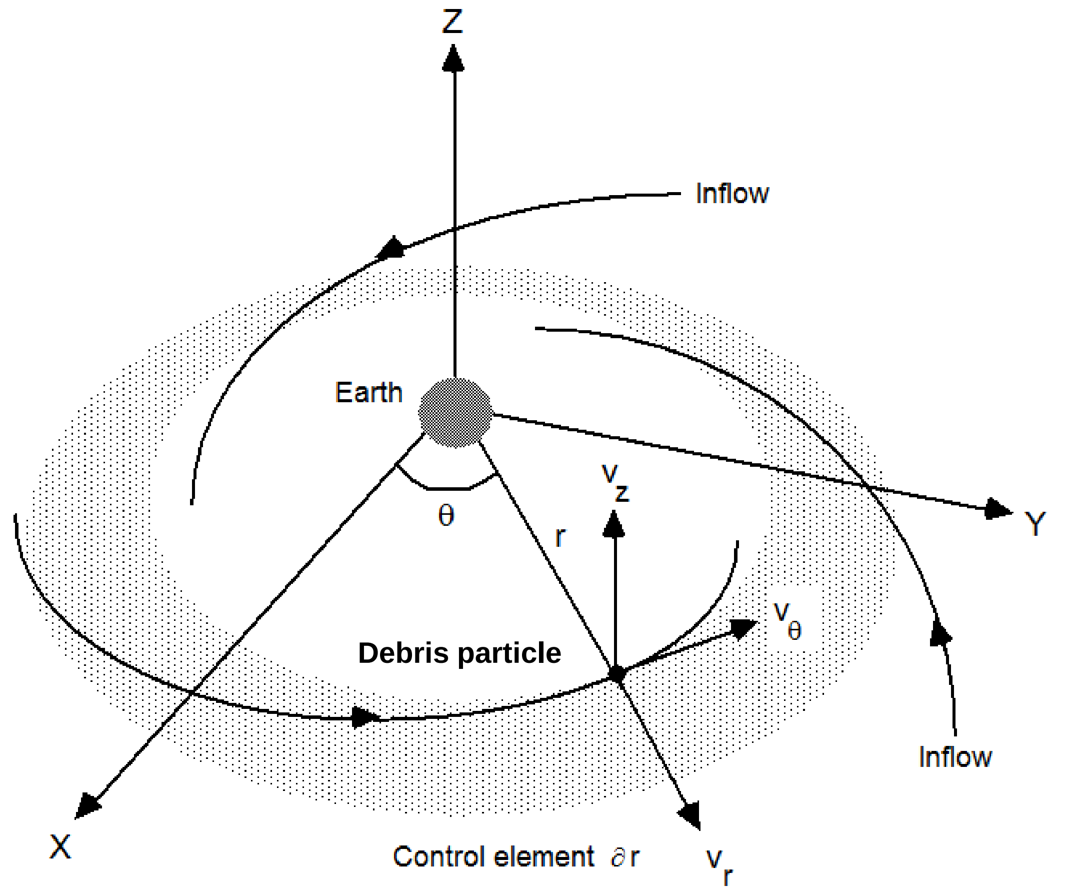
Debris inflow

- **Consider inflow of debris through control element at radius *r* and of width Δr**
- **In control volume/area $2\pi r\Delta r$ have a total number of particles $2\pi r\Delta r.n(r,t)$**
- **Need to conserve total number of particles – remember $v_r\,(r)$ is negative!**

$$\frac{\partial}{\partial t}\left(2\pi r\Delta r n(r,t)\right)=v_r(r)2\pi r n(r,t)$$ ***Flow in***

$$-v_r(r+\Delta r)2\pi(r+\Delta r)n(r+\Delta r,t)$$ ***Flow out***

$$+2\pi r\Delta r\left(\dot{n}^+(r,t)-\dot{n}^-(r,t)\right)$$ ***Deposition-Sweeper***


Z
Inflow
Earth
θ
r
v_z
Y
$v_θ$
Debris particle
Inflow
X
Control element ∂r
v_r

Continuity equation

- Now take the limit as control element width $\Delta r \to 0$ to obtain continuity PDE
- Solution describes time evolution of radial structure of the debris population
- Since we have a PDE need to provide some initial data n(r,0) defined at $t=0$

$$\boxed{\frac{\partial n(r,t)}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r(r) n(r,t)\right) = \dot{n}^{+}(r,t) - \dot{n}^{-}(r,t)}$$

$$\frac{\partial n(r,t)}{\partial t} + \frac{\partial n(r,t)}{\partial r} v_r(r) + n(r,t)\left[\frac{v_r'(r)}{v_r(r)} + \frac{1}{r}\right] v_r(r) = \dot{n}^{+}(r) - \dot{n}^{-}(r)$$

Solving the continuity equation

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- **Solution to ODE has arbitrary constant, solution to PDE arbitrary function . . .**
- **PDE is linear, so can in principle solve using the method of characteristics**
- **Separate single PDE into two ODEs – if can find characteristics can solve**

$$\left[-n(r,t)\left[\frac{v_r'(r)}{v_r(r)}+\frac{1}{r}\right]v_r(r)+\dot{n}^+(r)-\dot{n}^-(r)\right]^{-1}dn=v_r(r)^{-1}dr=dt$$

$$\frac{dr}{dt}=v_r(r)$$ ***Defines family of characteristics . . .***

$$\frac{dn(r,t)}{dr}+n(r,t)\left[\frac{v_r'(r)}{v_r(r)}+\frac{1}{r}\right]=\frac{1}{v_r(r)}\left[\dot{n}^+(r)-\dot{n}^-(r)\right]$$

Solving the characteristic equation

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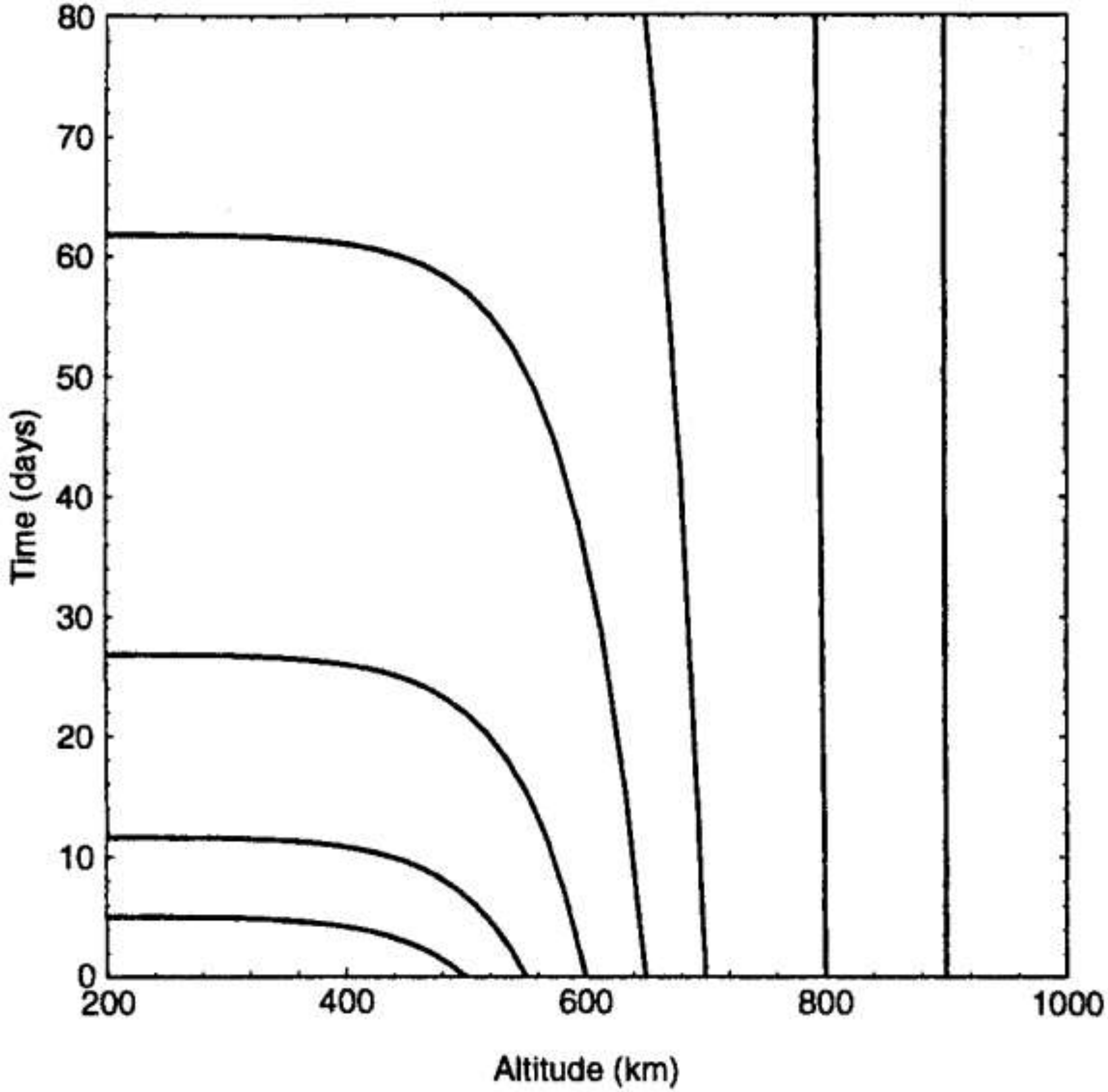
- **Can see that physically, the characteristic curve is the inwards flow of debris**
- **Integrate to obtain function relating orbit radius and time - the *characteristics***
- **Needs this to obtain the general solution to the continuity equation PDE later**

$$\frac{dr}{dt} = v_r(r) = -\sqrt{\mu r} B \rho_o \exp\left(-\frac{r-R}{H}\right)$$

$$\frac{1}{\alpha}\int \frac{\exp(r/H)}{\sqrt{r}} dr + t = -C \quad , \quad \alpha = \sqrt{\mu} B \rho_o \exp(R/H)$$

$$\boxed{\exp(r/H) + \frac{\alpha\sqrt{R}}{H} t = \tilde{C} \quad , \quad \tilde{C} = -\frac{\alpha\sqrt{R}}{H} C}$$

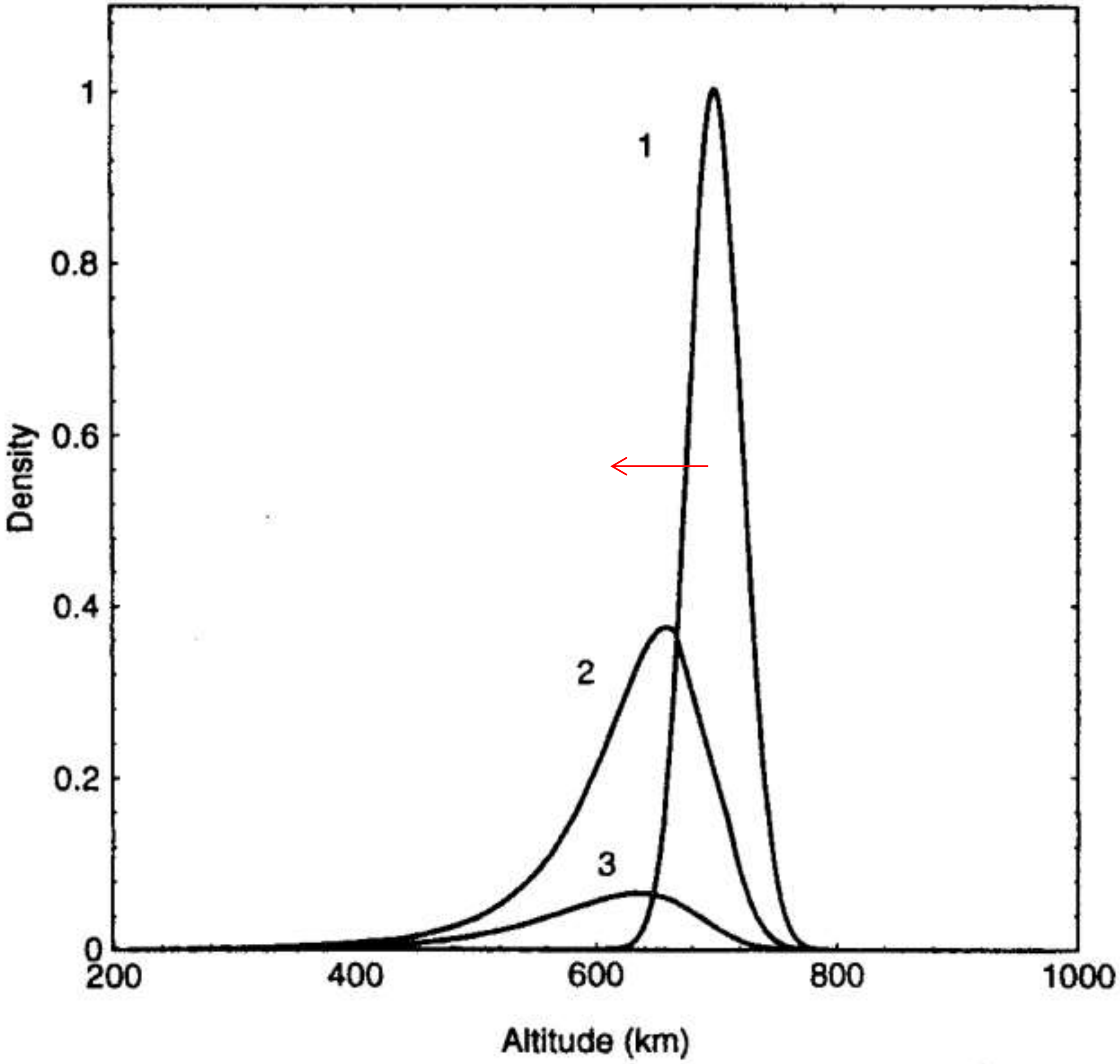

Time (days)
80
70
60
50
40
30
20
10
0
200
400
600
800
1000
Altitude (km)

Debris evolution

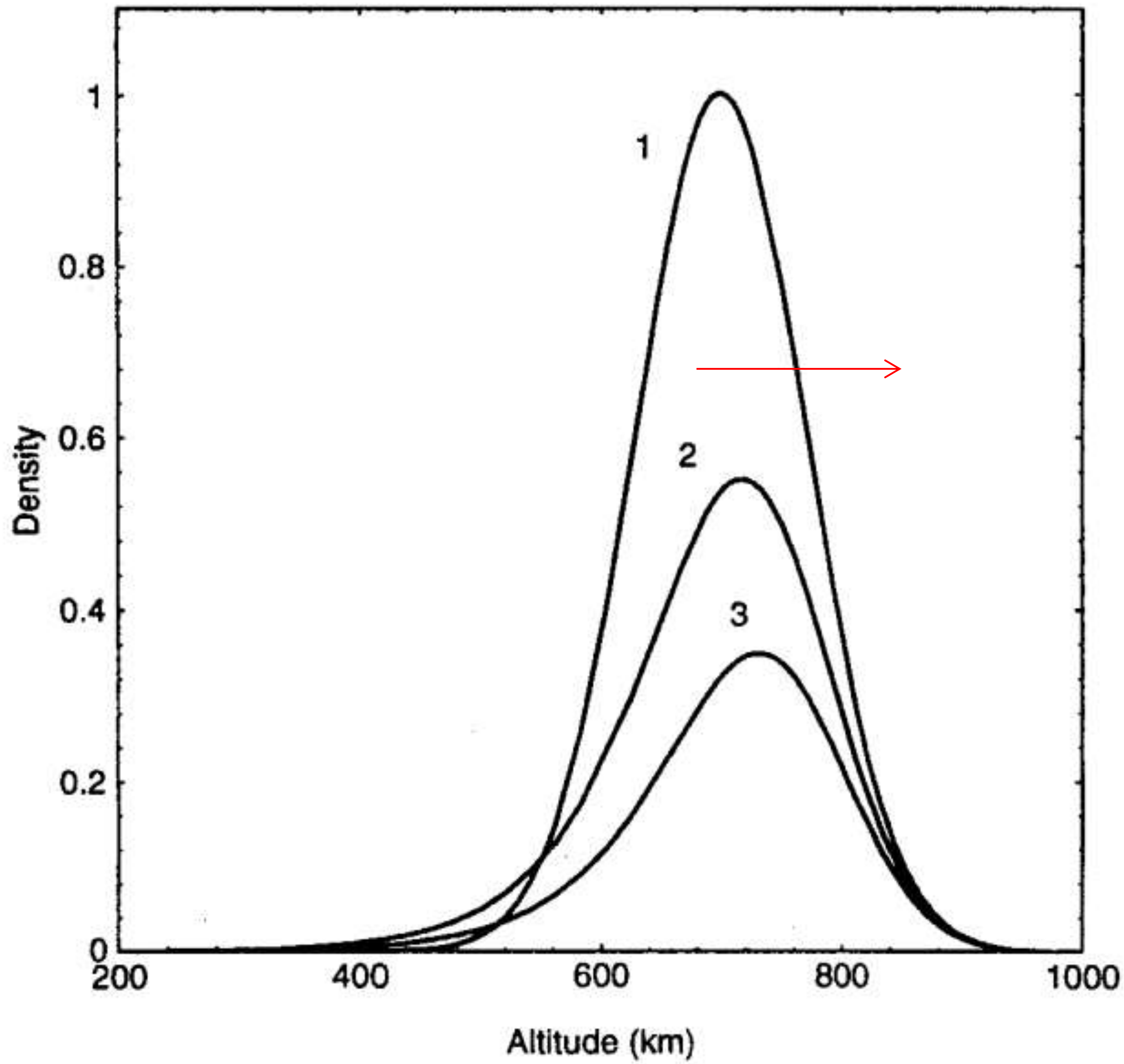
- **Assume that there is no deposition of new debris particles, or debris removal**
- **Simply ODE need to solve to obtain the debris density distribution function**
- **Add arbitrary function of characteristic equation – determine from initial data**

$$\frac{dn(r,t)}{dr} + n(r,t)\left[\frac{v_r'(r)}{v_r(r)} + \frac{1}{r}\right] = 0$$

$$\boxed{ln\{n(r,t)\} = ln\{rv_r(r)\}^{-1} + \Psi(G(r,t))}$$



1
2
3
Density
Altitude (km)
0
0.2
0.4
0.6
0.8
1
200
400
600
800
1000


1
2
3
Density
0
0.2
0.4
0.6
0.8
1
200
400
600
800
1000
Altitude (km)

Asymptotic distribution

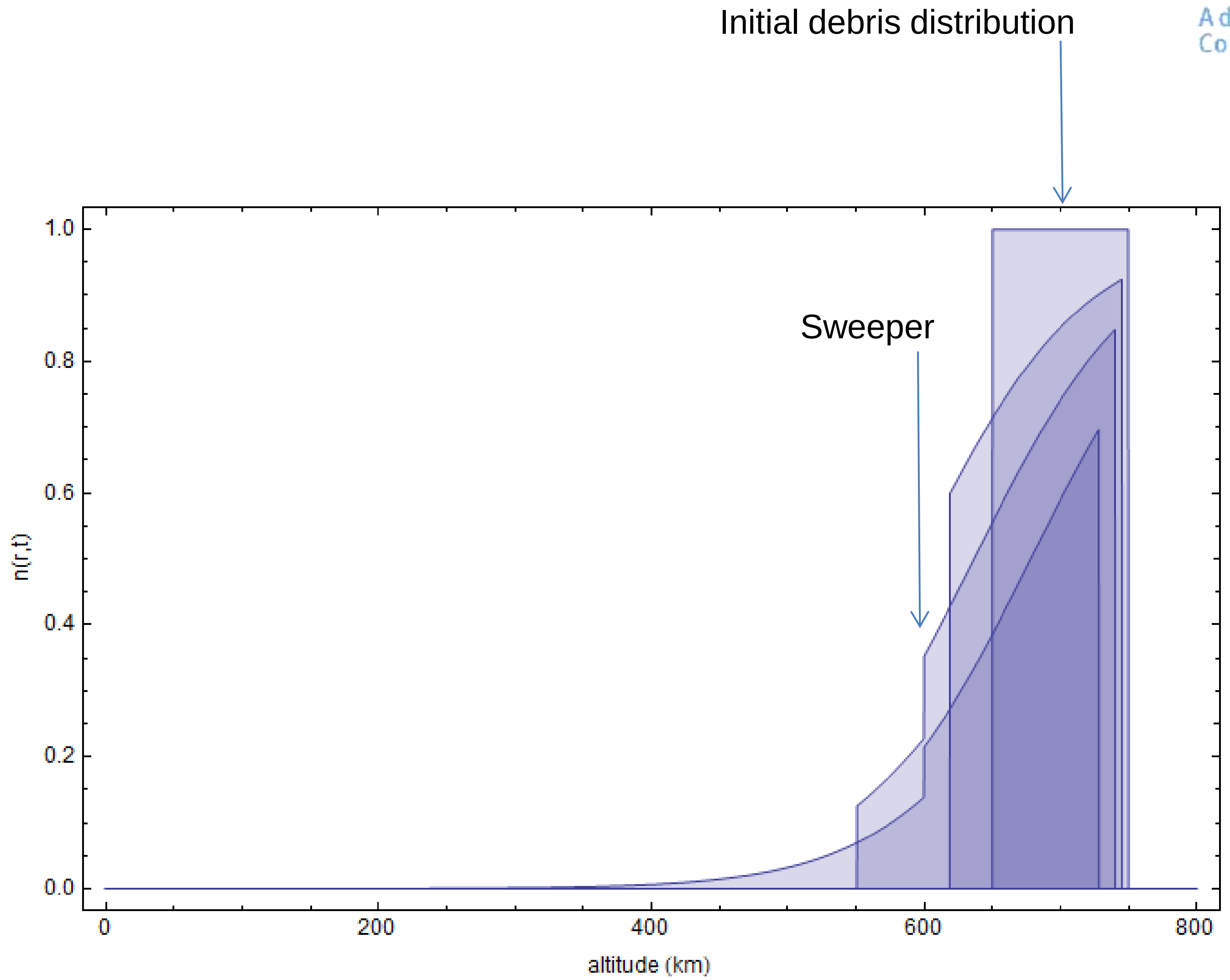
- **If we now have deposition of new debris particles then steady-state possible**
- **Deposition of new particles balanced by the removal of debris by air drag**
- **Find that the asymptotic solution of the continuity PDE given by an integral**

$$\boxed{n_\infty(r) = \frac{1}{r^2 v_r(r)} \int r^2 \dot{n}^+(r)\, dr + D}$$

Debris sweepers

- Now assume we have a debris sweeper device active on a fixed circular orbit
- Define using a Dirac delta function so device is active at single orbit radius
- Assume the sweeper removes fixed fraction of debris inflow passing it's orbit

$$\boxed{\dot{n}^{-}(r,t) = -2\pi r v_r(r) n(r,t) \lambda \delta(r - \bar{r})}$$


Initial debris distribution
Sweeper
1.0
0.8
0.6
0.4
0.2
0.0
n(r,t)
0
200
400
600
800
altitude (km)

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